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Go Figure 2004

For Students in grades 7, 8, 9, 10, 11, and 12

Show your work. You can receive partial credit for partial solutions. Please write all solutions clearly, concisely, and legibly.

The positive integers are the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\dots$

1. Assume that each person paints at a steady rate. Rates in doors per hour do not have to be integers.
   (a) Alice can paint a door in 20 minutes. How many doors can she paint in an hour?
   (b) Barbara can paint a door in 80 minutes. How many doors can she paint in an hour?
   (c) If they work together, how many doors can Alice and Barbara paint in an hour?
   (d) If they work together, how many hours willd it take for Alice and Barbara to paint 30 doors?

2. In this problem, each of A, B and C represents a different digit. For example, if F represents 6 then $3F$ represents 36 and $F4F$ represents 646. In each of these problems, the hidden digits are different from all other (not hidden) digits used in the problem. For each of the following problems, find digits A, B, and C that make the problem a correct statement.
   (a) $AA6 \times 7AB = 4C8CCB$.
   (b) $\frac{BAA}{63BA} = \frac{CA}{9C}$.

3. In an arithmetic progression, the difference $s - t$ of adjacent terms t, s is fixed. For example, the arithmetic progression $4, 8, 12, 16\dots$ has $(8-4) = (12-8) = (16-12) = 4$ as the fixed difference.
   (a) How many terms are in the arithmetic progression $8, 16, 24, 32, \dots, 2776$? [The three dots after the number 32 mean "and so on until."]
   (b) What is the value of the middle term in the arithmetic progression of part a?
   (c) How many terms are in the arithmetic progression $-20, -12, -4, 4, 12\dots, 2748$?
   (d) What is the value of the middle term in the arithmetic progression of part c?
   (e) What is the average value of the first term and the last term of the progression in part c?
   (f) What is the average value of the second term and the next-to-last term of the progression in part c?
   (g) What is the sum of the terms of the progression on part c?
   (h) How many numbers are the sum of a pair of different terms from the progression in part c?

4. Given integers x and y, the notation x^y means x multiplied by itself y times. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. By definition $x^0 = 1$ for all x.
   (a) What is 11^2? 11^3? and 11^4?
   (b) What is the units digit of 11^{342}? The units digit of a number is the value of the "ones" place. For example, the units digit of 547 is 7.
   (c) What is the units digit of $\frac{11^{1386}-1}{10}$?

5. For each of the following, find a value of x that makes the statement true.
   (a) $2 \times 2^3 = 2^x$.
   (b) $3^2 \times 3^3 = 3^x$.

(c) $2 \times 2^4 \times 2^5 = 2^x$.

(d) $10^0 \times 10^8 \times 10^{16} \times 10^{24} \cdots \times 10^{2768} = 10^x$.

6. We wish to color a set of indistinguishable balls. That is, before the balls are painted, they are all the same. Each ball receives one color and colors can be repeated. For example, if we have three balls and three colors (red, green, blue), then 2 blue balls with 1 red ball is a possible coloring. It does not matter which two balls are colored blue. This is different from the coloring with 2 red balls and 1 blue ball. For the case of 3 balls and 3 colors, there are 10 possible color combinations.
[Hint for the remainder of the problem: The number of distinct subsets with exactly k elements that can be chosen from a set of n elements is usually represented as $\binom{n}{k}$, and called "n choose k." We have $\binom{n}{0} = 1 = \binom{n}{n}$ and $\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r}$ for $r = 1, 2, \ldots, n-1$.]

   (a) How many ways can you color 3 balls with 4 colors?

   (b) How many ways can you color 3 balls with 500 colors?

7. (a) Find a three-digit integer K such that $(K^2 - 1)$ is a multiple of five and K does not have either 1 or 6 for a units digit.

   (b) How many four-digit integers K exist such that $K^3 + 1$ is a multiple of 7 but $K + 1$ is not a multiple of 7?

8. Consider the following game, which proceeds in rounds. At the beginning of the game there are ten people. Each receives $\frac{1}{10}$ of a pie. You must now select a nonempty subset of the people to leave the game (anywhere from 1 to 10 people). These people will give their pie to you and leave. For the next round, a new pie is divided equally among all the remaining people. For example, if there are five people left, each gets $\frac{1}{5}$ of a pie. You again select a nonempty subset to leave, and they give their pie to you at that time. The rounds continue until no people are left. Represent a strategy with a list of the number of people chosen for each round. The numbers in the list always sum to 10. For example, the strategy is $5, 3, 2$, means you select 5 people in the first round, 3 people in the second round, and 2 people in the last round. For this strategy, you will acquire $5 \times \frac{1}{10} + 3 \times \frac{1}{5} + 2 \times \frac{1}{2}$ pies. What strategy gives you the most pie at the end of the game?

9. Consider a set of 36 boxes arranged in 6 rows of 6 boxes each as shown in the figure. Two boxes are *neighbors* if they are next to each other (up-and-down, side-to-side, or diagonally). For example, in the figure, all the boxes labeled x are neighbors of the box labeled y.

		x	x	x	
		x	y	x	
		x	x	x	

Suppose we place the numbers $1, 2, \ldots, 36$ into the boxes, one per box. There are $36! = 36 \times 35 \times \ldots \times 2 \times 1$ ways to do this. The *gap* of a placement is the maximum difference between the values placed in any two neighbors. For example, in the following 4×4 example with the numbers $1, \ldots, 16$, the gap of the placement is 13 because neighboring boxes hold the numbers 2 and 15 in the lower left corner ($15 - 2 = 13$), and no other pair of neighbors has a bigger difference between their values.

4	11	9	3
10	1	13	14
2	12	5	8
15	6	16	7

What is the smallest possible gap among all ways to place the numbers $1, 2, \ldots, 36$ into the 6×6 set of boxes?

10. A set of integers S *produces* a number n if n is the sum of a subset of S with each element multipled by some positive integer. For example, if $S = \{3, 10\}$, then S produces $3 = 1 \times 3, 6 = 2 \times 3, 9 = 3 \times 3, 10 = 1 \times 10, 13 = 1 \times 3 + 1 \times 10, 16 = 2 \times 3 + 1 \times 10$, and so on. Because there is no limit on the size of the multipliers, any non-empty set produces an infinite number of values n. If $S = \{1\}$, then S produces all the positive integers since any positive integer n is equal to $n \times 1$. The *horizon* of a set S is the smallest integer h such that S produces *every* integer greater than or equal to h. For example, the horizon of $\{1\}$ is 1. It's possible that a set has no horizon. For example, set $\{3, 6\}$ has no horizon.

    (a) What is the horizon of set $\{2, 3\}$?

    (b) What is the horizon of set $\{3, 5\}$?

    (c) What is the horizon of set $\{21, 35\}$?

    (d) What is the smallest number n such that set $\{21, 35\}$ produces all multiples of 7 greater than or equal to n?

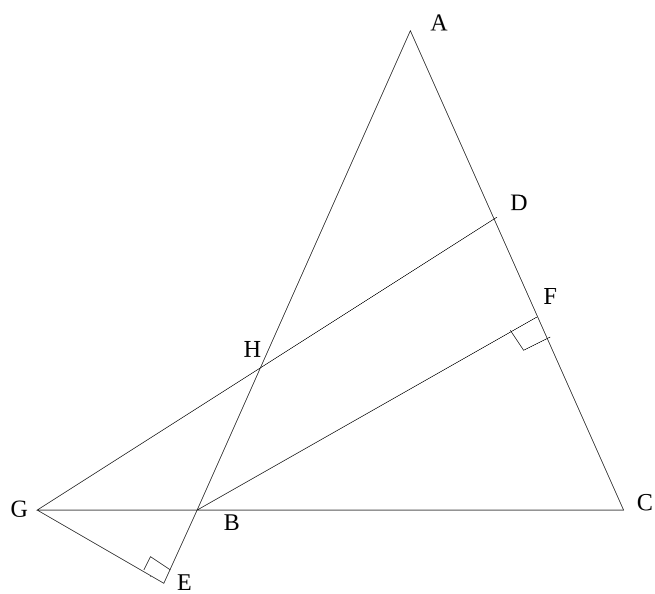
    (e) What is the horizon of set $\{21, 22, 35\}$?

    (f) What is the horizon of set $\{359, 5943, 6226, 9905\}$?

11. As clearly as you can, justify your answer for Problem 10f.

12. The figure for this problem is not drawn to scale. We are given $AB = AC$. BF is perpendicular to AC (makes a right angle). $AHBE$ is a straight line and it is perpendicular to GE. GD is parallel to BF. $GD = 16$, $GE = 4$, and $AB = 13$.

    (a) What is the length of segment BF? Hint: you might want to add a line segment at point B.



    (b) What is the length of segment AD?